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OPTIMUM DAMPING OF THE OSCILLATIONS OF A PENDULUM WITH UNCERTAIN EXTERNAL AND PARAMETRIC PERTURBATIONS†

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A control law for a parametrically excited pendulum, for which the level of damping of the oscillations takes a value that is close to the minimum possible value, is obtained by the methods of H_∞-control theory. The results of simulation are presented.

H_∞-control theory, which is lately being actively developed [1, 2], enables one to synthesize robust regulators for systems with uncertainty [3, 4]. The equations of such systems contain unknown parameters or functions which are, in particular, parametric perturbations.

One of the unsolved problems in H_∞-control theory is to find the lowest possible (over all permissible controls) level of damping of oscillations which is understood as the maximum (with respect to all external perturbations) of the ratio of the norm of the output of the system to the norm of the external perturbation. Mathematically, this problem is associated with the existence of a special solution of a parametric Riccati matrix equation which contains a number of parameters, one of which corresponds to the level of damping of oscillations in the system. Up to the present time, constructive conditions for the existence of such a solution have been lacking and the only possibility was to check using the MATLAB software package whether the required solution for a given level of damping of oscillations exists.

Estimates of the limits of the minimum possible level of damping of oscillations of a parametrically perturbed pendulum, obtained by solving a problem on the limiting possibilities of a control [6], are presented below. A robust H_∞-control of a pendulum is constructed which ensures a level of damping of oscillations which is close to the minimum possible value.

1. FORMULATION OF THE PROBLEM

Consider a controlled pendulum with parametric and external perturbations

$$\ddot{x} + \dot{x} + \omega_0^2[1 + f\Omega(t, x, \dot{x})]x = u + \upsilon, \quad x(0) = \dot{x}(0) = 0 \tag{1.1}$$

where ω_0, $f(\omega_0 \neq 0, 0 \leq f < 1)$ are specified parameters, μ is a control and $\upsilon = \upsilon(t)$ is an external perturbation. The dissipation factor is chosen to be equal to unity, which is ensured by a corresponding change of the independent variable t. The function $\Omega(t, x, \dot{x})$ defines the parametric perturbation and satisfies the condition

$$|\Omega(t, x, \dot{x})| \leq 1, \quad \forall t, x, \dot{x} \tag{1.2}$$

We will denote the class of such functions by Σ. With respect to the external perturbation $\upsilon(t)$, we shall assume that $\upsilon \in L_2(0, \infty)$, i.e.

$$J_1(\upsilon) = \int_0^\infty \upsilon^2(t)dt < \infty$$

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The class of permissible controls is determined by linear feedbacks of the type

$$u = -\alpha x - \beta \dot{x} \tag{1.3}$$

We will denote the class of such control laws by Ξ.

In order to describe the purpose of the control, we introduce a functional which characterizes the quality of a transient

$$J_2(u, \upsilon) = \int_0^{\infty} (\omega_0^2 x^2 + \dot{x}^2 + \rho^2 u^2) dt$$

where ρ is a specified parameter. The integrand of this functional corresponds to the mechanical energy of the unperturbed pendulum, taking account of that spent on the control.

The problem of damping the oscillations of a pendulum consists of determining a control from the class Ξ which ensures that the inequality

$$J_2(u, \upsilon)/J_1(\upsilon) < \gamma, \quad \forall \upsilon \in L_2, \quad \upsilon \not\equiv 0, \quad \forall \Omega(t, x, \dot{x}) \in \Sigma \tag{1.4}$$

is satisfied with the minimum possible value of the parameter $\gamma > 0$.

For a given permissible control law, we define the level of damping of oscillations in the system as follows

$$\Gamma(u) = \sup_{\Omega \in \Sigma} \sup_{\upsilon \not\equiv 0} [J_2(u, \upsilon)/J_1(\upsilon)]$$

In the set of permissible controls of the level of damping of oscillations in the system when there are external and parametric perturbations, we define the minimum possible control as

$$\gamma_0 = \inf_{u \in \Xi} \Gamma(u) \tag{1.5}$$

such that problem (1.4) is solvable for all $\gamma > \gamma_0$ and does not have a solution for all $\gamma \leqslant \gamma_0$. In control theory, this problem is known as a H_∞-control problem.

In order to solve it, we reduce Eq. (1.1) to the form

$$\dot{\mathbf{x}} = A_f(t, \mathbf{x})\mathbf{x} + B_1 \upsilon + B_2 u \tag{1.6}$$

in which

$$\mathbf{x} = \begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix}, \quad A_f(t, \mathbf{x}) = \begin{Vmatrix} 0 & 1 \\ -\omega_0^2[1 + f\Omega(t, \mathbf{x})] & -1 \end{Vmatrix}, \quad B_1 = B_2 = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}$$

2. SYNTHESIS OF A PENDULUM CONTROL WHEN THERE ARE NO PARAMETRIC PERTURBATIONS

We will initially consider a pendulum without parametric perturbation, that is, when $f = 0$. In this case, system (1.6) takes the form

$$\dot{\mathbf{x}} = A_0 \mathbf{x} + B_1 \upsilon + B_2 u \tag{2.1}$$

where

$$A_0 = \begin{Vmatrix} 0 & 1 \\ -\omega_0^2 & -1 \end{Vmatrix}$$

In accordance with relation (1.3), the permissible control laws have the form

$$u = -\Theta \mathbf{x}, \quad \Theta = (\alpha \beta) \tag{2.2}$$

As the controlling output, we introduce the vector

$$\mathbf{z} = C\mathbf{x} + Du, \quad C = \begin{Vmatrix} \omega_0 & 0 \\ 0 & 1 \\ 0 & 0 \end{Vmatrix}, \quad D = \begin{Vmatrix} 0 \\ 0 \\ \rho \end{Vmatrix} \tag{2.3}$$

and consider the problem of the H_∞-control of system (2.1), which consists of constructing a control law which, for a given value of $\gamma > 0$ accompanying the zero initial conditions $\mathbf{x}(0) = \mathbf{0}$ from any perturbation from the class $L_2(0, \infty)$, ensures that the following equality is satisfied:

$$\frac{\|\mathbf{z}\|}{\|\upsilon\|} < \gamma, \quad \forall \upsilon \in L_2, \quad \upsilon \not\equiv 0 \tag{2.4}$$

and, when there are no perturbation, the asymptotic stability of the closed system. Here, for any vector function $\mathbf{h}(t) \in L_2$,

$$\|\mathbf{h}\|^2 = \int_0^\infty |\mathbf{h}(t)|^2 dt$$

and $|\mathbf{h}|$ is the Euclidean norm. Hence, the problem of the damping of the oscillations of a pendulum is equivalent to the H_∞-control problem of system (2.1).

We will also present another treatment of this problem.

We introduce the notation

$$S_\upsilon = \sup_{\upsilon \not\equiv 0} \frac{\|\mathbf{z}\|}{\|\upsilon\|}, \quad I_u = \inf_{u \in \Xi} \frac{\|\mathbf{z}\|}{\|\upsilon\|}$$

For any permissible control, we set up, corresponding to system (2.1), (2.3), and operator H which maps a perturbation $\upsilon(t)$ from L_2 into an output $\mathbf{z}(t)$ from L_2. Problem (2.4) can then be formulated in the form

$$\|H\| < \gamma, \quad \|H\| = S_\upsilon$$

where $\|H\|$ is the norm of this operator.

Under the assumption that the closed system (2.1), (2.2) is asymptotically stable, we introduce the Laplace transforms $V(p)$ and $\mathbf{Z}(p)$ for $\upsilon(t)$ and $\mathbf{z}(t)$ respectively. Then

$$\mathbf{Z}(p) = H(p)V(p)$$

where the transfer matrix

$$H(p) = \frac{1}{p^2 + (1+\beta)p + (\omega_0^2 + \alpha)} \begin{Vmatrix} \omega_0 \\ p \\ -\rho(\alpha + \beta p) \end{Vmatrix}$$

Using the Parseval equality, it can be shown that

$$\|H\| = \|H\|_\infty = \sup_\omega \sqrt{H^T(-i\omega)H(i\omega)}$$

where the quantity $\|H\|_\infty$ is the H_∞-norm of the system being considered. Hence, in terms of the H_∞-theory of the level of damping of oscillations, $\Gamma(u)$ is identical to the H_∞-norm of the closed system in the case of this control law and the minimum possible level of damping of oscillations γ_0 is identical to the minimum H_∞-norm over all permissible control laws.

The solution of problem (2.4) follows from general H_∞-control theory [1]: one of the possible (the so-called central) H_∞-control law has the form

$$u = -\rho^{-2} B_2^T P\mathbf{x} \tag{2.5}$$

where $P \geqslant 0$ is the stabilizing solution of the Riccati matrix equation

$$A_0^T P + PA_0 + PBP + C^T C = 0, \quad B = \gamma^{-2} B_1 B_1^T - \rho^{-2} B_2 B_2^T \tag{2.6}$$

that is, the solution for which the matrix $A_0 + BP$ is a Hurwitz matrix. The Riccati equation is solved numerically using the MATLAB software package. Here, it can turn out that the required solution does not exist for the chosen value of the parameter γ, and the problem arises of determining the possible values of γ.

3. ESTIMATION OF THE MINIMUM LEVEL OF DAMPING OF OSCILLATIONS GENERATED BY EXTERNAL PERTURBATIONS

We will consider the problem of finding the minimum possible level of damping of oscillations γ_0 of a pendulum defined by expression (1.5) when there are no parametric perturbations. In this case, γ_0 is defined as follows:

$$\gamma_0 = \inf_{u \in \Xi} S_\upsilon \tag{3.1}$$

A direct calculation of this quantity is not possible. We will therefore attempt to obtain a lower estimate for it.

An approach, based on the analysis of the limiting possibilities for controlling a linear system [6], is proposed for constructing this estimate. The essence of this approach consists of treating an auxiliary maximin problem. We will use a relation; well known from game theory which relates a maximin and a minimax:

$$\inf_{u \in \Xi} S_\upsilon \geq \sup_{\upsilon \neq 0} I_u$$

If the quantity

$$\gamma_* = \sup_{\upsilon \neq 0} I_u \tag{3.2}$$

could now be found, it would be a lower estimate of the required quantity γ_0. In this sense, problem (3.2) can be regarded as a problem concerning the limiting possibilities of the control of system (2.1) when there are acting external perturbations from the class L_2.

It should be noted that problem (3.2) is, to a certain extent, simpler than the initial problem (3.1). In any case, for each specified perturbation $\upsilon(t)$, the problem of minimizing the quadratic functional can be effectively solved [7]. On the other hand, the result of the minimization, in the set of all permissible controls, of the ratio of the norm of the output of the system to the norm of the external perturbation cannot be written in the form of a simple expression containing $\upsilon(t)$ which is available for the subsequent analysis. We will therefore attempt to give an estimate of the form

$$\gamma_* = \sup_{\upsilon \neq 0} I_u \geq \gamma_+$$

We will first consider the problem of constructing a lower estimate of the minimum value of $\|\mathbf{z}\|$ with respect to $u \in \Xi$ for each specified $\upsilon(t)$. In order to do this, we specify a certain perturbation $\upsilon(t)$ from the class L_2, choose an arbitrary control $u(\mathbf{x}) = -\Theta\mathbf{x}$, which ensures the asymptotic stability of system (2.1), and obtain a lower estimate of $\|\mathbf{z}\|$. Suppose $\mathbf{x}(t)$ is the solution of the Cauchy problem for system (2.1) with zero initial conditions, a specified perturbation and a chosen control $u(\mathbf{x})$. We use the notation $\hat{u}(t) = u(\mathbf{x}(t))$ and obtain

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + B_1\upsilon(t) + B_2\hat{u}(t) \tag{3.3}$$

We now change to Fourier transforms in the last equality, by multiplying (3.3) by the factor $e^{-i\omega t}$ and integrating the resulting equation with respect to t within the limits from $-\infty$ to ∞. In Fourier transforms, we obtain

$$\begin{gathered} i\omega\mathbf{X} = A_0\mathbf{X} + B_1 V + B_2 U \\ \mathbf{X} = \int \mathbf{x}(t)e^{-i\omega t}dt, \quad V = \int \upsilon(t)e^{-i\omega t}dt, \quad U = \int \hat{u}(t)e^{-i\omega t}dt \end{gathered} \tag{3.4}$$

Integration with respect to t (and, subsequently, with respect to ω) is carried out from $-\infty$ to ∞.

We express the vector $\mathbf{X}$ from Eq. (3.4) as

$$\mathbf{X} = RB_1V + RB_2U, \quad R = (i\omega I - A_0)^{-1} \tag{3.5}$$

and, using the Parseval equality, we write the expression for the square of the norm of the output of the system

$$\|\mathbf{z}\|^2 = \frac{1}{2\pi}\int(\mathbf{X}^*C^TC\mathbf{X} + \rho^2U^*U)d\omega \tag{3.6}$$

where an asterisk denote Hermitian conjugation. Substituting the first expression of (3.5) the integrand on the right-hand side of equality (3.6), we obtain

$$\begin{aligned} \|\mathbf{z}\|^2 &= \frac{1}{2\pi}\int[K(|U|^2 - U^*U_0 - U_0^*U) + B_1^TLB_1|V|^2]d\omega \\ K &= B_2^TLB_2 + \rho^2, \quad L = R^*C^TCR, \quad U_0 = -K^{-1}B_2^TLB_1V \end{aligned} \tag{3.7}$$

We next consider the following auxiliary problem: it is required to find the function U, which minimizes the integrand in (3.7), for any specified function V (and, consequently, for any specified function $\upsilon(t)$).

Its solution is formulated as follows: the minimum value of the integrand with respect to U in relation (3.7) is reached when

$$U = U_0 = -K^{-1}B_2^TLB_1V \tag{3.8}$$

The proof of this assertion can be found in [6].

With U_0 from relation (3.8), the integrand in (3.7) is reduced to the form $G(\omega)\,|V|^2$, where

$$G(\omega) = B_1^T(L - LB_2K^{-1}B_2^TL)B_1 \tag{3.9}$$

We also note that, since the integrand in (3.6) is non-negative, then $G(\omega) \geqslant 0$.

So, for any specified perturbation $\upsilon(t)$ and for any fixed control from the class Ξ, we have the estimate

$$\|\mathbf{z}\|^2 \ge \frac{1}{2\pi}\int G(\omega)|V|^2d\omega$$

This means that

$$I_u \ge \int G(\omega)|V|^2d\omega/\int|V|^2d\omega$$

Next, no calculating sup over $\upsilon \in L_2$, we obtain

$$\sup_{\upsilon \neq 0} I_u \ge \sup_{V \neq 0}\left[\int G(\omega)|V|^2d\omega/\int|V|^2d\omega\right] = \max G(\omega)$$

Henceforth, unless otherwise stated, max is calculated with $\omega \in (-\infty, \infty)$.

Finally, the required estimate of the limiting possibilities of the control acquires the form

$$\gamma_0 \ge \gamma_* \ge \max\sqrt{G(\omega)} = \gamma_+ \tag{3.10}$$

For system (2.1) and (2.3), the function $G(\omega)$ is defined by the relation

$$G(\omega) = \frac{\rho^2(\omega^2 + \omega_0^2)}{\rho^2[(\omega^2 - \omega_0^2)^2 + \omega^2] + \omega^2 + \omega_0^2}$$

Maximization of this function with respect to ω gives the following estimate for the minimum possible level of damping of the oscillation of a pendulum when there are no parametric perturbations:

$$\gamma_+ = \rho/\sqrt{\rho^2\omega_0^2\theta^2 + 1}$$

where

$$\theta^2 = \begin{cases} 1, & \omega_0 \le \sqrt{3}/3 \\ \omega_0^{-2}(2\omega_0 - \sqrt{4\omega_0^2 - 1})\sqrt{4\omega_0^2 - 1}, & \omega_0 > \sqrt{3}/3 \end{cases}$$

For example, when $\rho = 1$ and $\omega_0 = 10$, a permissible control does not exist for which the level of damping of the oscillations of the pendulum is less the $\gamma_+ \approx 0.817$. On solving the Riccati equation (2.6) numerically with $\gamma = 0.819$, we find that the control law (2.5), which ensures a given level of damping of oscillations, has the form

$$u_1 = -0.50x_1 - 1.79x_2 \tag{3.11}$$

This mean that the minimum possible level of damping of the oscillations of the pendulum lies within the limits $0.817 \leqslant \gamma_0 \leqslant 0.819$.

4. SYNTHESIS OF A PENDULUM CONTROL IN THE CASE OF PARAMETRIC PERTURBATIONS

We will consider the problem of the damping of the oscillations of a pendulum in the case of parametric perturbations ($f \neq 0$) and external perturbations. From the point of view of H_∞-theory, problem (1.4) is the problem of a robust H_∞-control that consists of constructing a control law which, for a specified value of $\gamma > 0$ with null initial conditions $\mathbf{x}(0) = \mathbf{0}$ and for any perturbation $\upsilon(t)$ from the class $L_2(0, \infty)$ and any permissible parametric perturbation $\Omega(t, \mathbf{x})$ from the class Σ, ensures that the inequality

$$\frac{\|\mathbf{z}\|}{\|\upsilon\|} < \gamma, \quad \forall \upsilon \in L_2, \quad \upsilon \not\equiv 0, \quad \forall \Omega(t, \mathbf{x}) \in \Sigma \tag{4.1}$$

is satisfied and, in the absence of external perturbations, ensures the asymptotic stability of the closed system.

In order to solve this problem, we consider the auxiliary system

$$\dot{\mathbf{x}} = A_0\mathbf{x} + B_1\upsilon + F\xi + B_2u, \quad \mathbf{z} = C\mathbf{x} + Du \tag{4.2}$$

where $F = \mathrm{col}(0, f)$, ξ is an additional variable and all the remaining variables and parameters are the same as in the initial system (1.6). When $\xi = -\Omega(t, \mathbf{x})E\mathbf{x}$, where $E = (\omega_0^2, 0)$, Eqs (4.2) and (1.6) are identical and, in this case, it follows from condition (1.2) that

$$|\xi(t)| \le |y(t)|, \quad \forall t \ge 0, \quad y = E\mathbf{x} \tag{4.3}$$

We rewrite the first of Eqs (4.2) and introduce the new control output ($\hat{\mathbf{z}}$)

$$\dot{\mathbf{x}} = A_0\mathbf{x} + (B_1, \gamma\mu^{-1}F)\mathbf{w} + B_2u, \quad \hat{\mathbf{z}} = \mathrm{col}(\mathbf{z}, \mu y) \tag{4.4}$$

Here $\mathbf{w} = \mathrm{col}(w_1, w_2)$, $w_1 = \upsilon$, $w_2 = \gamma^{-1}\mu\xi$ and $\mu \neq 0$ is a certain parameter. The control law which ensures for system (4.4) that the target inequality

$$\|\hat{\mathbf{z}}\| < \gamma\|\mathbf{w}\|, \quad \forall \mathbf{w} \in L_2, \quad \mathbf{w} \not\equiv 0 \tag{4.5}$$

is satisfied for a certain value of μ will ensure that inequality (4.1) is satisfied for system (1.6) for the same value of the parameter γ. In fact, from inequality (4.5), we obtain

$$\|\mathbf{z}\|^2 < \gamma^2\|\upsilon\|^2 + \mu^2(\|\xi\|^2 - \|y\|^2)$$

and this means that inequality (4.1) will hold for system (1.6) for which the condition $\|\xi\|^2 - \|y\|^2 \leqslant 0$, which follows from condition (4.3), is satisfied. On the other hand, satisfying the target inequality (4.1) for system (1.6) with a certain control law still does not mean, generally speaking, that, in the case of the same control law and the same value of γ, a value of μ exists such that this control law ensures the attainment of the target (4.5) for system (4.4).

Hence, the central H_∞-law of control for the auxiliary, completely defined system (4.4) can be taken as a robust H_∞-control of a pendulum when there are parametric perturbations. This control law can have the form (2.5), where $P \geqslant 0$ is the stabilizing solution of the Riccati matrix equation

$$A_0^T P + PA_0 + P(B + \mu^{-2} FF^T)P + C^T C + \mu^2 E^T E = 0 \tag{4.6}$$

This parametric equation is also solved numerically using the MATLAB software package. It may turn out here that the required solution does not exist for the chosen values of the Parameters μ and γ. The question of separating a domain in the (μ, γ) plane in which Eq. (4.6) is solvable will be considered in the following section using the approach described in Section 3.

5. BOUNDARIES OF THE MINIMUM LEVEL OF DAMPING OF THE OSCILLATIONS OF A PENDULUM IN THE CASE OF PARAMETRIC PERTURBATIONS

We recall that the minimum possible level of damping of oscillations of a pendulum in the case of parametric perturbations is defined as follows:

$$\gamma_0 = \inf_{u \in \Xi} \Gamma(u) = \inf_{u \in \Xi} \sup_{\Omega \in \Sigma} S_v \tag{5.1}$$

so that inequality (4.1) is solvable for all $\gamma > \gamma_0$ and has no solution for all $\gamma \leqslant \gamma_0$. It is natural to call the quantity γ_0 the minimum robust H_∞-norm. We will now find boundaries of the interval within which γ_0 necessarily lies.

We will takes as the lower boundary γ_1 of the quantity γ_0, the minimum robust H_∞-norm of system (1.6) for a narrower class of parametric perturbations and, in fact, for the steady-state parametric perturbations $\Omega(t, \mathbf{x}) \equiv \Omega_0$ which satisfy (1.2). The quantity γ_1 can be directly obtained by maximizing with respect to Ω_0 the estimate (3.10) derived above

$$\gamma_0 \geq \gamma_l = \max_{\Omega_0} \max_{\omega \in (-\infty, \infty)} \sqrt{G(\omega, \Omega_0)} \tag{5.2}$$

The function $G(\omega, \Omega_0)$ is calculated for each specified Ω_0 in accordance with formula (3.9) and the quantity ω_0^2 in the matrix A_0 is replaced by $\omega_0^2(1 + f\Omega_0)$. As a result, we obtain

$$G(\omega, \Omega_0) = \frac{\rho^2(\omega^2 + \omega_0^2)}{\rho^2\{[\omega^2 - \omega_0^2(1 + f\Omega_0)]^2 + \omega^2\} + \omega^2 + \omega_0^2} \tag{5.3}$$

In order to construct the upper limit of the quantity γ_0, we make use of the auxiliary system (4.4). We introduce the notation

$$\hat{S}_w = \sup_{\mathbf{w} \neq 0} \frac{\|\hat{\mathbf{z}}\|}{\|\mathbf{w}\|}, \quad \hat{I}_u = \inf_{u \in \Xi} \frac{\|\hat{\mathbf{z}}\|}{\|\mathbf{w}\|}$$

For system (4.4), we define the minimum H_∞-norm, which depends on the parameters γ and μ:

$$\nu_0(\gamma, \mu) = \inf_{u \in \Xi} \hat{S}_w \tag{5.4}$$

It follows directly from inequality (4.5) that

$$\nu_0(\gamma, \mu) \leq \gamma \tag{5.5}$$

We will consider the following equation in γ

$$\nu_0(\gamma, \mu) = \gamma$$

We will assume that, when $\gamma > 0$, $\mu > 0$, this equation implicitly defines a function $\gamma = \gamma_*(\mu)$ such that, for a specified μ, inequality (4.5) is satisfied when $\gamma > \gamma_*(\mu)$ and is not satisfied when $\gamma \leqslant \gamma_*(\mu)$ (as will be evident from what follows, this situation holds in the case of a pendulum). Since the H_∞-control for the auxiliary system in the case of a given μ is the robust H_∞-control for the initial system with a parametric perturbation with the same value of γ, the inequality

$$\gamma_0 \le \gamma_*(\mu)$$

holds for the minimum robust H_∞-norm.

Then,

$$\gamma_u = \inf_{\mu \ne 0} \gamma_*(\mu) \tag{5.6}$$

can serve as the upper limit γ_u of the quantity γ_0.

Hence, the minimum robust H_∞-norm lies within the following limits

$$\gamma_l \le \gamma_0 \le \gamma_u \tag{5.7}$$

The quantity γ_1 is defined by expression (5.2).

The following computational procedure is required to construct the boundary γ_u. It consists of finding the pair of values ($\mu > 0, \gamma > 0$) for which a stabilizing solution P of the two-parameter Riccati equation (4.6) exists. The value of γ found can be taken as the upper limit γ_u. It may turn out to be a rough estimate, and it is therefore desirable to find a smaller value of γ if possible.

We will next show how it is possible to narrow down the domain in which the required parameters are searched for in the set $\mu > 0$, $\gamma > 0$. Here, the basic idea consists of the "rejection" of a certain region of the first quadrant in which a stabilizing solution of the Riccati equation (4.6) clearly does not exist.

For this purpose, we will first apply to system (4.4) the approach described above when obtaining the estimate (3.10). Actually, in the case of fixed γ and μ for this system, we obtain

$$\nu_0(\gamma, \mu) \ge \sup_{\mathbf{w} \ne 0} \hat{I}_u \ge \sup_{\mathbf{W} \ne 0} \left(\frac{\int \mathbf{W}^* G_u(\omega) \mathbf{W} d\omega}{\int |\mathbf{W}|^2 d\omega} \right)^{1/2} = \nu_+(\gamma, \mu)$$

where

$$G_u(\omega) = \left\| \begin{matrix} B_1^T \\ \gamma\mu^{-1} F^T \end{matrix} \right\| [L_u - L_u B_2 K_u^{-1} B_2^T L_u](B_1, \gamma\mu^{-1} F)$$

$$K_u = B_2^T L_u B_2 + \rho^2, \quad L_u = R^*(C^T C + \mu^2 E^T E) R, \quad R = (i\omega I - A_0)^{-1}$$

$$\nu_+(\gamma, \mu) = \max \sqrt{r(G_u(\omega))}$$

$\mathbf{W}$ is the Fourier transform of the function $\mathbf{w}(t)$ and $r(\cdot)$ is the spectral radius of the corresponding matrix. In the case being considered, we have

$$G_u(\omega) = \Phi(\omega) \left\| \begin{matrix} 1 & \mu^{-1}\gamma f \\ \mu^{-1}\gamma f & \mu^{-2}\gamma^2 f^2 \end{matrix} \right\|$$

and the spectral radius of this matrix is equal to

$$r(G_u(\omega)) = (1 + \mu^{-2}\gamma^2 f^2)\Phi(\omega)$$

where

$$\Phi(\omega) = \frac{\rho^2[\omega^2 + \omega_0^2(1 + \mu^2\omega_0^2)]}{\rho^2[(\omega^2 - \omega_0^2)^2 + \omega^2] + \omega^2 + \omega_0^2(1 + \mu^2\omega_0^2)}$$

We now consider the following equation in γ

$$\nu_+(\gamma, \mu) = \gamma \tag{5.8}$$

the solution of which we denote by $\gamma_+(\mu)$. Then, the Riccati equation (4.6) clearly does not have the required solution in the domain $\gamma \leqslant \gamma_+(\mu)$.

We note that

$$\max\sqrt{r(G_u(\omega))} = (1+\mu^{-2}\gamma^2 f^2)^{1/2}\Psi(\mu), \quad \Psi(\mu) = \max\sqrt{\Phi(\omega)}$$

and the solution of Eq. (5.8) can be explicitly expressed in the form

$$\gamma_+^2(\mu) = \frac{\mu^2\Psi^2(\mu)}{\mu^2 - f^2\Psi^2(\mu)}$$

In the narrowed-down region $\gamma > \gamma_+(\mu)$, we then choose μ with a certain step size and, for each such μ, we find numerically (using the MATLAB software package, for example) the minimum possible value of γ for which the Riccati-equation (4.6) has a stabilizing solution. The minimum value from the values of γ which have been obtained is then chosen and this value is also taken as the upper estimate for γ_u.

We emphasize once again that the lower and upper estimates in inequality (5.7) are found using the numerical procedures which have been presented, and that methods which exist at the present time do not enable one, in general, to find the exact value of γ_0 and the robust control corresponding to it. In this connection, a design procedure for constructing a robust H_∞-control can be presented as follows. As was shown above, a H_∞-control with a specified level γ of damping of perturbations for the auxiliary system (4.4) is a robust H_∞-control for the initial uncertain system (1.5) with the same γ. Consequently, the control law

$$u = -\rho^{-2}B_2^T P\mathbf{x} \tag{5.9}$$

can be chosen as the required robust H_∞-control, where $P \geqslant 0$ is the stabilizing solution of Eq. (4.6) in the case of a μ from the domain of definition of the function $\gamma_*(\mu)$ and when $\gamma > \gamma_*(\mu)$. It should also be noted that the level of damping of the perturbations in the uncertain system (1.5) cannot be made smaller than the lower estimate γ_l which has been obtained for any permissible control law.

We will present these boundaries in the case of a pendulum with $\omega_0 = 10, f = 0.05, \rho = 1$. The lower boundary γ_l is determined numerically according to (5.2) and (5.3) : $\gamma_l \approx 0.820$. Numerical analysis shows that the minimum value of the function $\gamma_+(\mu)$ is reached when $\mu = 0.141$ and is equal to 0.942. We take $\gamma_u \approx 0.943$. Hence

$$0.820 \leq \gamma_0 \leq 0.943$$

Solving the Riccati equation (4.6) numerically with $\mu = 0.141$ and $\gamma = 0.943$, we find the robust control (5.9) in the form

$$u_2 = -1.503x_1 - 3.955x_2 \tag{5.10}$$

6. SIMULATION

By means of simulation, we compare the damping of the oscillations of a parametrically perturbed pendulum in the case of two strategies: the control law (3.11), constructed for a parametrically unperturbed pendulum, and the robust control law (5.10). The simulation was carried out for the following external and parametric perturbations:

$$v(t) = \begin{cases} \sin 10t, & 0 \leq t \leq 20 \\ 0, & t > 20 \end{cases}, \quad \Omega(t, \mathbf{x}) = \sin 20t$$

(the frequencies of the perturbations are close to resonance frequencies). As a result, in the case of control law (3.11) the ratio of the norm of the output to the norm of the external perturbation was found to be equal to 1.150, and in the case of the control law (5.10) was found to be equal to 0.903. This means that, the case of the given control laws, the levels of damping of oscillations satisfy the inequalities $\Gamma(u_1) \geqslant 1.150$ and $\Gamma(u_2) \geqslant 0.903$. On the other hand, according to the procedure for constructing a robust H_∞-control, we have $\Gamma(u_2) \leqslant 0.903$. Consequently, the control law u_2 ensures a level of damping of the oscillations $\Gamma(u_2)$ within the limits of 0.903 to 0.943 in the case of parametric and external perturbations. For comparison, we point out that, when there is no control, ($u = u_0 = 0$) $\Gamma(u_0) \geqslant 1.753$, which exceeds the minimum possible level of damping of oscillations by a factor of at least 1.86. Note that, in the case of the control laws, u_0 and u_1, the upper boundaries in the parametrically perturbed systems cannot be indicated in principle.

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REFERENCES

1. DOYLE, J. C., GLOVER, K., KHARGONEKAR, P. P. and FRANCIS, B. A., State-space solutions to standard H_2 and H_∞ control problems. *IEEE Trans. Automat. Control.*, 1989, **34**, 8, 831–847.
2. BARABANOV, A. Ye. and PERVOZVANSKII, A. A., Optimization by frequency-uniform criteria. *Avtomatika i Telemekhanika*, 1992. **9**, 3–32.
3. XIE, L. and DE SOUZA, C. E., Robust H_∞ control for linear systems with norm-bounded time-varying uncertainty. *IEEE Trans. Automat. Control.*, 1992, **37**, 8, 1188–1191.
4. KOGAN, M. M., Synthesis of robust H_∞ – suboptimal regulators as a solution of a differential game under conditions of uncertainty: direct and inverse problems. *Avtomekhanika i Telemekhanika*, 2000, 7, 109–120.
5. KWAKERNAAK, H., Robust control and H_∞-optimization. *Automat.* 1993, **29**, 2, 255–273.
6. BALANDIN, D. V., Limiting possibilities for controlling linear systems and estimating the minimal H_∞-norm. *Izv. Ross. Akad. Nauk, Teor. i Sistemy Upravleniya*, 2001, 6, 50–56.
7. SALUKVADZE, M. Ye., The analytical construction of an optimal regulator for persistent perturbations. *Avtomatika i Telemekhanika*, 1962, 6, 721–731.

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